Option Pricing with Two Risky Assets and Financial Leverage

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Abstract[π]

This study presents an alternative option pricing theory to the Black-Scholes model in which the balance sheet is composed of both short-term and fixed assets and by both long-term debt and equity. The resulting model is used to examine the volatility smile. It is demonstrated that the smile characteristics can be produced by choices of the parameters of the model. Hang Seng Index data is then used to investigate whether actual index and option prices can be compatible with the model's specifications.

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Option pricing under the lognormal distributional assumptions of the Black-Scholes model is well known to exhibit bias across simultaneous exercise prices. This gives rise to the characteristic known as the "smile," describing the pattern of implied volatilities of the observed prices of the options under their range of exercise prices. Numerous efforts have been made to correct this bias, notably the stochastic volatility of Hull and White, and the GARCH modelling of Duan, as two classic models that suggested that the constancy of the only unobservable parameter, the return variance, was an unrealistic assumption. An alternative explanation presumes that the lognormal distribution, as a continuous process in space, is invalid as a description of the stock return. This leads then to a model based on a jump distribution, which offers other possibilities of explaining option prices without creating a volatility smile; for an early example, see Cox and Ross.

The appeal of the lognormal distribution lies in its mathematical tractability and reasonable fit to observed stock returns over longer periods. Two particular instances of this include the compound option pricing model of Geske, and the displaced distribution model of Rubinstein. Geske's model, describing an option on an option, allows the equity of a firm to be considered as and option on the outstanding debt of the firm. Rubinstein extended this and decomposed the asset structure of the firm into those of relatively high and those of relatively low risk; for example, "fixed" assets including some very intangible assets such as intellectual capital produce cash flows of quite uncertain outcome, whereas the current assets have a relatively fixed rate of return barring non-payment and inventory mark-downs.

Rubinstein established his model by analogy to the Black-Scholes process of creating a risk-free hedge based on the stochastic value of the stock return. He argued that risk neutrality could be established provided that both assets were separately tradable. At the same time, the converse proposition is that if the two assets cannot be separated, with only their combined value traded as the single stock price, then a risk neutral valuation could not be established. For this reason, he posited a risk-free value for the low-risk asset which led to a displacement of the underlying diffusion price, and hence his "displaced diffusion" model.

We have re-examined Rubinstein's argument and extended it to design a model for a firm having a traditional balance sheet. This provides a combination of both low- and high-risk assets on the left-hand side (essentially current and fixed), and both long-term debt and equity on the right-hand side. In effect, the current assets represent net working capital, transferring current liabilities from the right- to the left-hand side. While it is impossible to argue that both assets are actually traded, it is common to see both accounts receivable and inventories traded as assets given the current practice of securitizing them in the short-term market; furthermore, it is reasonable to regard the cash portion as non-stochastic. Hence one can argue that the tradability of the component low-risk assets and the independently traded equity imply a net tradability for the fixed asset value.

In this examination, we re-derive our extension of Rubinstein's model and produce an expression for the value of an option on the traditional firm, under the assumption of independence of the returns of low- and high-risk assets. This expression includes parameters for the initial values of the low- and high-risk assets, as well as their individual variances, and the level of long-term debt (assumed riskless), in addition to the standard parameters of exercise price, riskless rate and time to maturity. Essentially, this has increased the number of unobserved parameters from one to two; however, in the context of an empirical investigation, the relative size of low- to high-risk assets and the level of debt are taken as unknown. The resulting four parameters can be manipulated to demonstrate that option prices consistent with a smile for a single variance can be explained by a fixed set of the four parameters. Trying different combinations of parameters permits the creation of a number of different smile patterns, such as have been observed in the past from options data.

We also examine index data to test the ability of the model to fit actual prices. This requires an automated approach of searching for parameters that enable the modelled price to fit to observed prices across the exercise price spectrum. The approach involves a learning process of determining the combined effects of varying parameters in raising or lowering the calculated price at different ends of the spectrum, followed by a fitting process that minimizes the bias in the result. Data is originally taken from a sample of U.S. data for the S&P 500 index for the learning process. The data for the testing is taken from the Hong Kong and Singapore market indexes for the year 2000.????

In terms of the hypothesized model, this approach should be used for individual companies with their actual, although not perfectly observable, balance sheet structures. The two variance parameters would then be estimated from the option price observations. Alternatively, the five parameters can be varied for a best-fit solution, and the results compared with the actual balance sheet data. In the case of index data, the output of parameter estimates must be compared with actual estimates of the average capital structure of the component firms listed in the index; similarly, the asset ratio would have to be assessed for reasonableness against the ratios of component firms. Another test of validity of the model is stability of the estimates across time, especially of the estimated balance sheet parameters. While the variance estimates can be expected to vary in time, as for the conventional estimates of Black-Scholes implied volatility, the balance sheet should remain relatively stable.

In the next section, we give a detailed review of the relevant literature and its relevance to this study. Then we present the model and develop the formula for the special case of uncorrelated assets. Following that, we produce some simulated results for a set of standardized parameters that demonstrate a variety of patterns as parameters are varied individually. Finally, we present some preliminary results from some index price observations and show the estimated parameters.

1. The Smile in Literature

Empirical studies of the Black-Scholes option pricing model and its variants have repeatedly revealed pricing biases that are inconsistent with the theoretical foundations of the model. Alternatives to the standard specification of the underlying diffusion process, including volatility corrections and jump processes or mixed diffusion-jump processes, have proven to be no more successful than the Black-Scholes model in matching market prices. The phenomenon of bias in the volatilities implied by market prices of options on the same asset, rather than a single volatility across the range of exercise prices (and maturities), has come to be known as the "volatility smile" due to the shape or pattern of its graph. Empirical and theoretical research on the smile has been one of the major areas of financial research, since the phenomenon was identified. Rubinstein (1985) tested alternatively specified models, which were unable to eliminate the smile. More recently, Rubinstein (1994) and Jackwerth-Rubinstein (1995) have developed an alternative that they claimed to be successful in explaining the market's post 1987 smile. They have shown that the implied risk-neutral distribution revealed by the market prices can be linked to a binomial tree, thus preserving the property of risk-neutral valuation. Besides this important theoretical point is the empirical finding that the implied distribution is not lognormal; instead, the implied distribution exhibits both kurtosis and skewness, and frequently is bimodal.

In the two Rubinstein papers, the major computational problem was the efficient recovery of the implied distribution from the data. The non-parametric approach matched results with the most general distribution needed to fit the data, but this required a lengthy programming algorithm to identify it. Significantly, independent research (Masson and Perrakis 2000) has shown that for large samples sizes, even a completely general distribution failed to satisfy all the observations; approximately forty percent of the sample events (one set of contemporaneous option exercise price quotes) have at least one pair of quotes that cannot be satisfied by any distribution fitting the rest of the set. (If one or more modelled price falling outside the quotes is treated as bad data or mis-pricing, then this percentage drops.)

Ritchey (1990) has proposed a model based on the family of non-Gaussian probability distributions known as the finite mixture of lognormal distributions, which he claimed could explain the exercise price biases in implied volatility. A mixture of even a small number of lognormals is sufficiently flexible to generate positively skewed, kurtotic, fat-tailed and multimodal distributions consistent with a number of posterior probability distributions derived by Rubinstein and Jackwerth-Rubinstein. The closed form of this model permits a parametric search for the distribution that would fit observed option index quotations. In addition, the mixture of two lognormals can be explained by two plausible descriptions of market structure. If market participants are acknowledged to comprise two groups, described as bulls and bears, with their relative proportions being the mixing variable, then the combined probability distribution for the market return will be the weighted sum of their individual, lognormal expectations. Alternatively, all investors could behave as if they believed in two scenarios of a normal market return (above the long run average) and a pessimistic result of a poor and even extremely negative return, both of these also having lognormal distributions. (This latter rationalisation is consistent with Rubinstein's speculation of investors' acting to hedge against a relatively catastrophic event following the 1987 crash, at the same time as participating for normal gains, thereby explaining the post-1987 smile pattern.)

Ritchey models the asset price distribution as the mixture:

L(S,T) = ë L_1 + (1 - ë) L_2

where L_j is the lognormal density with parameters ì $_j$ and ó$_j$. To be consistent with risk-neutral pricing, he restricts the means such that the weighted average mean for L is the riskless rate:

ë ì $_1$ + (1 - ë)ì $_2$ = r

Then the resulting option price can be found due to linearity of the integral operator as:

C = ë C_1 + (1 - ë) C_2

where each of the C_j 's is a modified Black-Scholes formula. (Applying this model to a stock index option, each of the ì j's would be reduced by q, the continuous dividend yield.)

The model proposed in this paper is consistent with the Ritchey model as it is based on the conclusion that investors perceive the actual terminal stock return distribution not to be lognormal, as evidenced by the Rubinstein (1994) paper. The lognormal has been used extensively in financial models due to three attractive aspects: it is analytically convenient, empirically good as an approximation to realised returns, and economically logical in giving proportional returns and limited liability. For this reason, the model assumes that although asset returns are more complex than simple lognormals, they are generated as combinations of lognormally distributed components. The resultant distributions can be shown to be sufficiently flexible to reproduce even the bimodal shape that Rubinstein derived and thus to be equally consistent with the market observations. The analytical representations of the distributions used by the model, however, are far more efficient to use in fitting to the data, in addition to having economic foundations.

Unlike Rubinstein's approach, which simply deduces what must have been the perceived distribution by investors, this model offers economic models to justify the derivation of such perceived distributions. For this reason, the parameters that are determined have economic significance and should have stability or a pattern of stable evolution over time. This desired property constitutes a testable hypothesis and gives purpose to the empirical research conducted. Besides testing the claim that the model can fit observed data as well as the arbitrary Rubinstein distribution, we will be determining whether individual companies have characteristics consistent with the model parameters in one case,

and whether the aggregate market perceptions could be consistent in the second.

2. The Balance Sheet Model

We identify two classes of assets as being fixed assets, having typical business risk, and financial assets, having relatively low risk. Financial leverage can then be incorporated by using the balance sheet relationship:

$$U(t) + V(t) = L(t) + S(t) \qquad (1)$$

where $U(t)$ = market value of fixed assets (per share)

$V(t)$ = market value of net working capital (per share)

$L(t)$ = market value of long-term debt (per share) - assumed riskless

S(t) = market value of equity (per share)

Corresponding to Rubinstein's a, we define

$$a = \frac{U(t)}{U(t)+V(t)} \tag{2}$$

as the current proportion of fixed assets to total assets. Corresponding to Rubinstein's b, we define the current capital structure of the firm by the debt/equity ratio,

$$b = \frac{L(t)}{S(t)}\,. \tag{3}$$

Note that this differs from Rubinstein only by assuming asset *V* to be relatively low risk, instead of riskless. We assume that the market values of both assets assume lognormal processes, or:

$$dU(t) = m_1 U(t)dt + s_1 U(t)dz_1\,, \tag{4}$$

$$dV(t) = m_2 V(t)dt + s_2 V(t)dz_2\,. \tag{5}$$

Consequently, the stock price follows a displaced diffusion process:

$$\begin{aligned} dS(t) &= dU(t) + dV(t) - dL(t) \\ &= \left[m_1 U(t) + m_2 V(t) - rL(t)\right]dt + s_1 U(t)dz_1 + s_2 V(t)dz_2\,, \end{aligned} \tag{6}$$

which is no longer lognormal.

Denote by

$$C[U(t), V(t), L(t), t;\, X, T] \tag{7}$$

the price of an European call option at time t, with strike price X and expiration time T. The initial time t is assumed to be 0, and we suppress the time index t whenever the context is clear. Given the two independent sources of uncertainty in the stock price process we require a hedge portfolio composed of two call options held long and the underlying asset held short. The value of the portfolio at time t can be expressed as

$$P = aC^A + bC^B - S\,, \tag{8}$$

where A and B are two arbitrary call options on the same asset, with differing strike prices, and a and b are their shares in the portfolio.

By Ito's formula, the instantaneous change of the value of the portfolio can be expressed as

$$
\begin{aligned}
dP = {} & \boldsymbol{a}\left(C_t^A + m_1 U C_U^A + m_2 V C_V^A + rLC_L^A + \tfrac{1}{2}\boldsymbol{s}_1^2 U^2 C_{UU}^A + \tfrac{1}{2}\boldsymbol{s}_2^2 V^2 C_{VV}^A + \boldsymbol{r}\boldsymbol{s}_1\boldsymbol{s}_2 UVC_{UV}^A\right)dt \\
& + \boldsymbol{b}\left(C_t^B + m_1 U C_U^B + m_2 V C_V^B + rLC_L^B + \tfrac{1}{2}\boldsymbol{s}_1^2 U^2 C_{UU}^B + \tfrac{1}{2}\boldsymbol{s}_2^2 V^2 C_{VV}^B + \boldsymbol{r}\boldsymbol{s}_1\boldsymbol{s}_2 UVC_{UV}^B\right)dt \\
& - \left(m_1 U + m_2 V - rL\right)dt \\
& + \left(\boldsymbol{as}_1 UC_U^A + \boldsymbol{bs}_1 UC_U^B - \boldsymbol{s}_1 U\right)dz_1 \\
& + \left(\boldsymbol{as}_2 VC_V^A + \boldsymbol{bs}_2 VC_V^B - \boldsymbol{s}_2 V\right)dz_2 ,
\end{aligned} \tag{9}
$$

where C_U^A is partial derivative of call A with respect to U (and similarly for the others), and $\boldsymbol{r}$ is the correlation coefficient of the two processes. Let $\boldsymbol{a}$ and $\boldsymbol{b}$ be chosen such that

$$\boldsymbol{a}C_U^A + \boldsymbol{b}C_U^B = 1, \tag{10}$$

and

$$\boldsymbol{a}C_V^A + \boldsymbol{b}C_V^B = 1, \tag{11}$$

then the portfolio is not only riskless (as the coefficients of the last two terms vanish), but also independent of the drifts of both processes (as the coefficients of :1 and :2 vanish in the first three terms). Using the Black-Scholes argument, the portfolio earns a riskless return

$$dP / dt = rP \, ;$$

hence

$$
\begin{aligned}
& \boldsymbol{a}\left(C_t^A + rLC_L^A + \tfrac{1}{2}\boldsymbol{s}_1^2 U^2 C_{UU}^A + \tfrac{1}{2}\boldsymbol{s}_2^2 V^2 C_{VV}^A + \boldsymbol{r}\boldsymbol{s}_1\boldsymbol{s}_2 UVC_{UV}^A\right) \\
& + \boldsymbol{b}\left(C_t^B + rLC_L^B + \tfrac{1}{2}\boldsymbol{s}_1^2 U^2 C_{UU}^B + \tfrac{1}{2}\boldsymbol{s}_2^2 V^2 C_{VV}^B + \boldsymbol{r}\boldsymbol{s}_1\boldsymbol{s}_2 UVC_{UV}^B\right) + rL \\
& = r\left(\boldsymbol{a}C^A + \boldsymbol{b}C^B - S\right).
\end{aligned} \tag{12}
$$

Denoting

$$F(A) = C_t^A + rLC_L^A + \tfrac{1}{2}\boldsymbol{s}_1^2 U^2 C_{UU}^A + \tfrac{1}{2}\boldsymbol{s}_2^2 V^2 C_{VV}^A + \boldsymbol{r}\boldsymbol{s}_1\boldsymbol{s}_2 UVC_{UV}^A - rC^A \, , \tag{13}$$

$$F(B) = C_t^B + rLC_L^B + \tfrac{1}{2}\boldsymbol{s}_1^2 U^2 C_{UU}^B + \tfrac{1}{2}\boldsymbol{s}_2^2 V^2 C_{VV}^B + \boldsymbol{r}\boldsymbol{s}_1\boldsymbol{s}_2 UVC_{UV}^B - rC^B \, ; \tag{14}$$

(12) can be re-expressed as

$$\boldsymbol{a}F(A) + \boldsymbol{b}F(B) = -r(S + L) = -r(U + V) \, , \tag{15}$$

where the last equality uses the balance sheet equation (1).

By a well-known duality argument (a version of Farkas' Lemma), for linear equations (10), (11) and (15) to be consistent, there must exist scalars (η_1, η_2,1) such that

$$\begin{aligned} \eta_1 C_U^A + \eta_2 C_V^A + F(A) &= 0, \\ \eta_1 C_U^B + \eta_2 C_V^B + F(B) &= 0, \\ \eta_1 + \eta_2 - r(U+V) &= 0. \end{aligned} \tag{16}$$

Since (16) applies to any pair of options, the scalars, η_1 and η_2, cannot depend on any option-specific parameters, such as the strike price and the time to expiration. One immediate solution by inspection to this system[1] is

$$\begin{aligned} \eta_1 &= rU, \\ \eta_2 &= rV, \end{aligned} \tag{17}$$

Since the first two equations of (16) hold for any option, we drop the superscripts, substitute for F(B), and re-express as a partial differential equation (PDE):

$$C_t + rUC_U + rVC_V + rLC_L + \tfrac{1}{2}\sigma_1^2 U^2 C_{UU} + \tfrac{1}{2}\sigma_2^2 V^2 C_{VV} + \rho\sigma_1\sigma_2 UVC_{UV} - rC = 0, \tag{18}$$

As a result, PDE (18) becomes preference-free, or risk-neutral, and the displaced diffusion option pricing with two risky assets is considered solved up to this point. The precise equilibrium option prices can be obtained from solving the PDE, subject to relevant boundary conditions, by many standard numerical procedures, or from taking a relevant expectation with a risk-neutral joint distribution.

3. A solution for uncorrelated asset processes and simulated example

If the two processes are uncorrelated, the conditional distribution of the stock price can be expressed as a convolution of two lognormal distributions:

$$f(S_T, T; S, 0) = \int_0 g(S_T + L_T - V_T) * h(V_T)\, dV_T\,, \tag{19}$$

where

$$\begin{aligned} g(U_T, T; U, 0) &= \frac{1}{U_T \sigma_1 \sqrt{2\pi T}} \exp -\frac{\left[\ln U_T - \ln U - (r - \sigma_1^2/2)T\right]^2}{2\sigma_1^2 T}\,, \\ h(V_T, T; V, 0) &= \frac{1}{V_T \sigma_2 \sqrt{2\pi T}} \exp -\frac{\left[\ln V_T - \ln V - (r - \sigma_2^2/2)T\right]^2}{2\sigma_2^2 T}\,. \end{aligned} \tag{20}$$

Following Cox and Ross (1976), the solution to (18) can be found by a risk-neutral valuation:

$$C = e^{-rT} \int_{-\infty}^{\infty} \max(0, S_T - X) f(S_T, T; S, 0) dS_T \,. \qquad (21)$$

Since the convolution cannot be expressed in a closed form, the risk-neutral valuation has to be numerically integrated. Nevertheless, the integration is reasonably simple. Since $U_T \geq 0$, $V_T \geq 0$, and $L_T = e^{rT} L$, the integration region can be divided into two parts, such that the option price can be found as

$$\begin{aligned}
& C(U, V, L, X, T; r, \sigma_1, \sigma_2) = e^{-rT} \int_0^{\infty}\int_0^{\infty} \max(0, U_T + V_T - e^{rT}L - X) g(U_T) h(V_T) dU_T dV_T \\
& = \int_0^{X+e^{rT}L} e^{-rT} \left\{ \int_{X+e^{rT}L-V_T}^{\infty} (U_T + V_T - e^{rT}L - X) g(U_T) dU_T \right\} h(V_T) dV_T \\
& + e^{-rT} \int_{X+e^{rT}L}^{\infty} \left\{ \int_0^{\infty} (U_T + V_T - e^{rT}L - X) g(U_T) dU_T \right\} h(V_T) dV_T \qquad (24) \\
& = \int_0^{X+e^{rT}L} \left[UN(d_1) - [(X - V_T)e^{-rT} + L] N(d_2) \right] h(V_T) dV_T \\
& + VN(d_3) + [U - (L + e^{-rT}X)] N(d_4),
\end{aligned}$$

where

$$\begin{aligned}
d_1 &= \frac{\ln[U / (X + e^{rT}L - V_T)] + (r + \sigma_1^2 / 2)T}{\sigma_1 \sqrt{T}}, \\
d_2 &= d_1 - \sigma_1 \sqrt{T}, \\
d_3 &= \frac{\ln[V / (X + e^{rT}L)] + (r + \sigma_2^2 / 2)T}{\sigma_2 \sqrt{T}}, \qquad (25) \\
d_4 &= d_3 - \sigma_2 \sqrt{T},
\end{aligned}$$

and $N(\;)$ denotes the normal distribution function. Hence, the solution involves only one numerical integration over V_T, and the integrand is just the Black-Scholes formula with U being the asset price and $X + e^{rT}L - V_T$ being the striking price.

Table 1 presents some representative call values with S = \$100, r = 0.05, $\sigma_1 = 0.2$. The cases with $\sigma_2 = 0$ correspond to Rubinstein's displaced diffusion model with only one risky asset, while our model is evaluated with $\sigma_2 = 0.05$. The table shows the call values with various combinations of the debt/equity ratio *b*, and the high risk asset proportion *a*, and cross a wide range of the striking prices and two times to expiration, T =

0.25 year and T = 0.5 year. The cases with b = 0 and a = 1 correspond to the Black-Scholes formula.

For a more convenient comparison, we convert all the call values into the implied volatilities (IV's) by the Black-Scholes formula, and present the results in Table 2. First note that the IV's do not vary with the time to expiration under Rubinstein's and our models. Second, when high-risk assets U represent a larger proportion $(a = 0.75)$, both models have similar IV's, regardless of the debt/equity ratio. It is also noticeable that both models exhibit the "typical postcrash smile" as documented by Rubinstein (1994, pp. 777), i.e., the IV is decreasing in the strike price. Third, the two models differ quite significantly when asset U counts for a smaller proportion $(a = 0.25)$. As the proportion of low-risk assets V increases, steeper negatively-sloped smiles occur, in contrast to Rubinstein's flatter and even positively-sloped smiles. Also, for **b** = 0 (no leverage), I.V. was increasing in exercise price at **a** =.75 (high fixed assets), but decreasing at **a** = .25; for **b** = 2 (high leverage), the smile reversed, decreasing at **a** =.75 and increasing at **a** = .25.Notice that, with b = 0, both models exhibit a similar "U-shaped" volatility smile. With b = 1, Rubinstein's volatility smile rises with strike prices, while ours stays almost constant. With b = 2, Rubinstein's smile continues to rise with the striking price, but our smile reverses to the post-crash smile.

It is clear that our model provides more variety of smiles than the Rubinstein model, but neither model is sufficient to explain the empirically documented steep post-crash smiles. One direction for improvement is to try various values of the correlation coefficient in future research, which will involve more complexity in numerical integration, but is computationally workable. It is worth noting that our argument about the aggregate tradability supporting the risk neutrality does not depend on the lognormality of the underlying asset processes. Therefore, it is possible that some alternative processes may fit the observed volatility smiles better than the lognormal processes.

4. Empirical Testing

Calibration of the Model

The preliminary stage of empirical testing involves a learning process to determine the response across the exercise price spectrum to changes in the model parameters. Although the response can be determined analytically for simple changes (for instance, the call value is obviously rising in the variance parameters and in the debt ratio), the magnitude of price changes to individual parameter variation and also in combinations causes an analytical approach to become fairly intractable; this is further exacerbated by examining the response of the implied volatility values, given that they are inverse functions of expression (24). It is also noteworthy that the normal volatility smile for a single return variance (a two-dimensional graph) is replaced by a smile in two variance parameters, requiring a different method of dramatization for the biases.

In order to initiate the calibration of the parameters, a few sampled data points from U.S. index options have been used. The process involved setting a representative parameter for the market dividend yield (initially 3%), then pre-selecting a starting value for the variance of the stock by inversion of the Black-Scholes formula for a given call price (at the applicable time to maturity, corresponding risk-free rate, index level and closest to-the-money exercise price). The total stock variance can be decomposed into independent Φ_1 and Φ_2 (whose weighted sum of squares equals the total variance given independence of the assets), with Φ_2 = .035 as a reasonable estimate of low-risk volatility[2]; Φ_1 is then determined from the equation

$$s^2 = a^2 s_1^2 t + (1-a)^2 s_2^2 \tag{26}$$

(The null valuation corresponds to parameters a and b at values of one and zero, respectively.) Parameters 1-a and b were then varied from .05 to .5 by increments of .05 and the modeled call price was estimated from expressions (24) and (25), modified to include a dividend stream. The immediate result was to detect the variation in the call price in response to the parameter variation. Since the initially calibrated value was for no debt and no low-risk assets, increasing the parameters immediately increased the observed call price from 27.904 to the values presented in Table 3. The table also shows the value of Φ_1 changing with parameter a to retain the total variance at the B-S implied variance level. The immediate significance of this is to demonstrate that the model produces differing call prices, even though the stock variance changes (as is predicted from the model).

The next step was to force the call price to remain constant at the initial (observed) value for the same set of variations in parameters a and b. This requires that overall variance and Φ_1 change to compensate for increasing or decreasing leverage and the asset structure. The values of these are presented in Table 4A and 4B. So far, this merely confirms that the observed price can be produced by any of the discrete pairs of parameters (a,b), or any interpolated values; note that there remains freedom to adjust the low-risk parameter Φ_2 from its initial value of .035, with corresponding (a,b) values. The test is then to calculate from the model the values of the call with alternative exercise prices, using the same assumed values for the parameters, and to compare these with the observations. In this case, the starting call had X=425 and the second call had X=400. Table 5A presents the value of the second call, where the call price is seen to be increasing again in b but increasing and then decreasing in 1-a. Table 5B presents call values for X=450, where the call is again increasing uniformly in b, but decreasing and then increasing in 1-a, after an initial slight increase (from 1-a = .05 to 1-a = .10).

The immediate observation is that the model gives clear opportunities to create smiles using actual data. Unfortunately, after fitting a and b parameters to the first call, neither of the alternative calls were trading at values within the ranges produced, under the assumed values for Φ_2 and the risk-free rate and dividend yield. These also can be varied, noting that the risk-free rate and dividend yield must be consistent with observable values. This negative finding is neither conclusive nor fatal. Relaxing trade prices to points within the bid-asked spreads increases flexibility dramatically. Furthermore, the preliminary approach was to choose for some parameters such as ó$_2$, values that were intuitively appealing, but not necessarily correct.

The next stage of calibration consisted of using the entire data set for the first time point (being one time window on a single day). The data consisted of 18 simultaneous exercise price options of 3-months (74 days), 11 of 6-months (165 days), 3 of 9-months (256 days), and 4 of 1-year (347 days); the exercise prices ranged from 375 to 460 (by 5 for the 3-month), with an index level of 436.96, and applicable interest rates rising from .032 to .035 across the maturities.

For this data set, the implied B-S volatility was calculated from the closest option at each maturity, being either 435 for the shorter options and 425 for the longer two. This volatility was then used to find the B-S price across the exercise spectrum for each maturity (as a means of defining the pricing error for comparison with our modelled price error, in contrast to using a volatility smile as evidence of error). The smile phenomenon was apparent, as the one-year, four-option set at 375, 400, 425, 450 had decreasing volatilities of .1676, .1571, .1468, .1345, respectively. These implied volatilities were also used as starting points for the parametric pricing. At this point, it became apparent that for both models, the pricing error was reduced if the assumed dividend yield was reduced, and subsequently a rate of $q = .01$ was used.

Due to the use of implied volatility as a starting point, the programming uses sigma as a parameter and then determines Φ_1 from expression (26). Hence the effective parameter variation is in combinations of the Φ, a and b values. The first result of scanning the parameter ranges gave a fit of all four one-year options between their bid-asked ranges (but not **at** their mid-points); this success was greatly enhanced by finding that the parameter combination also fit the 9-month bid-asked ranges, even with the same Φ. Because these results were not consistent with the shorter and wider-ranging options, the eighteen 3-month options were used to find the least biased set of parameters with respect to the market prices. Then these parameters were tried for the other three maturity sets. The results are portrayed in Figures 1-4, where the model and B-S pricing errors (model – market) are graphed across the exercise prices.

In Figure 1, the one-year bias is apparent in the B-S pricing, under-pricing the market by $1.80 in the money, and over-pricing by $2 out of the money. (One should recognise the volatility smile equivalent, which would be negatively sloped.) The parametric model, on the other hand, produces a $.50 to .$25 over-pricing at each end with a $.40 under-pricing at the money. By any standard, it clearly prices better. Figure 2 shows over-pricing by both models at each end with slight under-pricing at the money by the parametric model, but the average fit is slightly improved over the B-S model. Figures 3 and 4 each show U-shaped pricing errors for both models, with the parametric model giving an improved fit in each case.

Figure 4, reflecting the shortest and most actively traded option series, bears the closest examination. Better fits are possible if the in-the-money option results are ignored; note that there tends to be the least trading and most mis-pricing in well in-the-money calls. The key aspect of Figure 4 is the 416 to 460 range, that centres on the index price.[3] The parametric model is less under- and over-priced, and essentially with equal error around zero. Trying to fit this region revealed the potential of the model. Evidently, removing the effects of a and b and using the implied volatility gives the B-S result. Increasing a and b and decreasing sigma together tends to raise the slope of the price, i.e., raising prices for high X and lowering them for low X. This means that when the pricing error is positively sloped, the model has limited ability to correct the market pricing bias relative to B-S. A negative slope, however, can easily be reversed. Most of the data points seen reveal a positive pricing error slope around the money, as in these examples, equivalent to a negatively-sloped volatility smile.

Despite this limitation, the results are extremely encouraging, in that they show that the "best fit" can be obtained with a consistent parameter set. The variation in the volatility is minimal over the four time periods; while a constant sigma is implied by the B-S lognormal distributional assumption (barring a GARCH or stochastic volatility model), it is rarely in practice observed, so the slight variation in sigma is tolerable. More crucial to the model is that the asset and liability structure is not open to much change, even over a period of successive days of observations. Hence the finding that the "best fit" was produced by exactly the same a and b parameters is significant. Also, the values used, debt ratio b = .27 and CA ratio of .175 are certainly reasonable numbers. (Note that "best fit" is not claiming that a better fit by more fine tuning is not achievable; but this would almost certainly lead to slightly different values for different maturities.)

In Figure 5 the actual three-month volatility smile is presented together with the implied volatilities that would be generated *if the market traded at the model price*. This demonstrates how a B-S analysis would react to a market in which the assumed criteria of this model were accepted by the market; that is, *if* the market were sensitive to asset mix and capital structure and priced by the model, a simple B-S analysis would presume a bias of the type shown in Figure 5. (Note that to create a flat smile for the model price, it would be necessary to have a constant volatility across the range, which would occur if the model degenerated to B-S, with a and b parameters not present.)

Hang Seng Index Options

Following the calibration, the model has been applied to Hong Kong data based on the Hang Seng Index (HSI) options and futures. The data consists of all intra-day call and futures trading for the year 2000. Although the model has not been fully automated, a preliminary analysis of an HSI trading window has been conducted, with a full analysis of multiple events to follow. (Volume processing will involve a computerized, intelligent search reflecting the learned responses to combined parameter changes, and probably using identified parameter values as seeds for the next data points.)

The HSI data is fairly thin for longer term options, so the first sample drawn used one-month calls (actually 18 days until expiry). Note that for the S&P data above, the longer term options with smaller exercise price sets was easier to fit; this was not necessarily a matter of the number of exercise prices, since the pricing was fairly consistent, but rather a case of having more adaptability for longer periods.

Because of the nature of the data, the first test used a window of about three minutes (from 9:53:45 to 9:57:09) during which twenty futures trades occurred between 15870 and 15925 (with implied spots of 15854 to 15894, with a slight upward direction in the last seven trades and fifty seconds; these last trades were associated with three call trades. The data set generated nine call trades with seven different exercise prices (of the same maturity). These were subdivided into four linked index and call price observations, but the nine trades were then graphed collectively. (A tenth observation with an eighth exercise price appeared to be anomalously over-priced in the market.) Figure 6 shows a potential solution for a sigma of .36 with debt ratio of .12 and C.A. ratio of .06; it also shows a sigma of .28 with debt ratio of .42 and C.A. ratio of .46, which is more steeply tilted. The higher sigma solution is preferable as it reduces the slope; other variations produced identical patterns which could be raised or lowered but not sloped less. This implies that the market pricing bias evidenced by the B-S prices cannot be eliminated. It alos strongly suggests that the market pricing of the 15200 and 16600 calls is significantly and anomalously low compared to the others.

We will be searching for other instances where a negative pricing bias is apparent and trying to correct for this. Once more individual cases are analysed, we will proceed to an automated determination of the optimal parameter combinations (**a**, **b**, $ó_1$, $ó_2$) in response to inputs of market prices. A number of search procedures are considered for this phase, including the Cyclic Coordinate Method, an unconstrained minimisation technique (for feasibility with respect to the lower and upper bounds on the price provided by the market quotes) and constrained optimisation by Newton, quasi-Newton and Levenberg-Marquardt methods.

Fitting the price model to the market quotations is accomplished by finding a feasible solution to the n upper and lower bounds on the model price at each exercise price imposed by the quotations. This can be accomplished as a nonlinear programming feasibility problem, but also by minimising the Euclidean distance norm in n dimensions; here the norm expresses the difference between the midpoints of the quotations and the modelled prices at each exercise price. If the minimum provides a feasible solution to the original problem then it is finished; if not, it will identify the closest point to feasibility and admit perturbations to consider almost feasible solutions (missing in one or more quotations). An efficient search methodology can be selected from the class of unconstrained nonlinear optimisation techniques or of constrained optimisation techniques.

5. Conclusion

The results of successful estimation of parameters must subsequently be analysed to assess the plausibility of the underlying model. We consider supportive evidence to be:

i) stability of the parameters throughout a trading day for sampled values

ii) stability of the parameters over successive days for an extended period, choosing a standard time during the trading day (such as 10:00 or 3:00)

iii) evidence of any common trend over time in parameters

iv) consistency of determined parameters and the actual market averages

The stability and the fourth property are the key questions to us. Instability would give poor support to the approach, although a general stability with aberrational values at random intervals would not be an unreasonable standard to expect. Given stability, the real test is whether the actual asset and debt ratios for the sample period were close to the estimated ratios in effect. The limited results so far for the S&P data are very encouraging, but not to be taken too seriously without corroboration over longer periods. The Hang Seng evidence is less encouraging, but appears to be due to the apparent mis-pricing in the sample used.

In addition, further theoretical work would need to examine the effect of correlation between the returns to fixed and financial assets, this requiring an extensively greater effort for numerical integration. The assumption to date of independence between the current and fixed assets does not appear to be unreasonable.

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Endnotes

[1] A more general solution is

$$\boldsymbol{h}_1 = (r - q_1)U, \qquad (17^*)$$
$$\boldsymbol{h}_2 = (r - q_2)V,$$

where the arbitrary q_1 and q_2 bear the interpretation as the "hypothetical dividend yields" that would enable assets *U* and *V* to be treated as if they are tradable assets in a risk-neutral world (see Hull (1989), Appendix 7D for details).

Equation (18) is then replaced by

$$C_t + (r - q_1)UC_U + (r - q_2)VC_V + rLC_L$$
$$+\tfrac{1}{2}\boldsymbol{s}_1^2U^2C_{UU} + \tfrac{1}{2}\boldsymbol{s}_2^2V^2C_{VV} + \boldsymbol{r}\boldsymbol{s}_1\boldsymbol{s}_2UVC_{UV} - rC = 0, \qquad (18^*)$$

which is identical to the last equation of Hull (1989, Appendix 7D, p. 180). However, the two dividend yields are not entirely arbitrary, because the third equation of (16), resulting from the aggregate tradability, restricts q_1 and q_2 such that

$$q_1U + q_2V = 0\,. \qquad (19^*)$$

Since *U* and *V* are both positive, q_1 and q_2 must assume opposite signs. This implies that, if one risky asset needs to pay a positive dividend to enter the risk-neutral world, the other asset must pay a negative dividend. No economic justification for this implication is apparent. Besides, even if a negative dividend is justifiable, it is impossible to find a pair of dividend yield functions to satisfy (19*), unless

$$q_1 = -q_2\frac{V}{U}\,; \qquad (20^*)$$

Such a restriction on q_1 based on *V* has no justification; hence rather than choosing a pair of non-zero dividend yields to satisfy (19*), it seems natural to set them both to zero.

Table 1
Representative Displaced Diffusion Call Values
S = \$100, r = 0.05, ó$_1$ = 0.2.

Panel A: Debt/Equity Ratio b = 0														
	T = 0.25							T = .5						
	a = 1	a = .75		a = .5		a = .25		a = 1	a = .75		a = .5		a = .25	
X	Black-Scholes	Φ_2=0	Φ_2=.05	Φ_2=0	Φ_2=.05	Φ_2=0	Φ_2=.05	Black-Scholes	Φ_2=0	Φ_2=.05	Φ_2=0	Φ_2=.05	Φ_2=0	Φ_2=.05
80	21.02	20.99	20.99	20.99	20.99	20.99	21.08	22.17	22.00	22.00	21.98	21.98	21.98	22.27
90	11.67	11.28	11.28	11.13	11.13	11.12	11.12	13.50	12.70	12.71	12.28	12.30	12.22	12.22
100	4.61	3.63	3.64	2.65	2.71	1.72	1.95	6.89	5.51	5.52	4.16	4.24	2.92	3.23
110	1.19	0.55	0.56	0.13	0.14	0.00	0.01	2.91	1.71	1.72	0.68	0.72	0.06	0.13
120	0.20	0.04	0.04	0.00	0.00	0.00	0.00	1.02	0.39	0.39	0.06	0.06	0.00	0.00
Panel B: Debt/Equity Ratio b = 1														
	T = 0.25							T = .5						
	a = 1	a = .75		a = .5		a = .25		a = 1	a = .75		a = .5		a = .25	
X		Φ_2=0	Φ_2=.05	Φ_2=0	Φ_2=.05	Φ_2=0	Φ_2=.05		Φ_2=0	Φ_2=.05	Φ_2=0	Φ_2=.05	Φ_2=0	Φ_2=.05
80		21.39	21.40	21.02	21.03	20.99	20.99		23.27	23.29	22.17	22.23	21.98	21.98
90		12.94	12.96	11.67	11.74	11.13	11.19		15.64	15.67	13.50	13.64	12.28	12.49
100		6.60	6.62	4.61	4.74	2.65	3.15		9.68	9.71	6.89	7.07	4.16	4.86
110		2.78	2.79	1.19	1.27	0.13	0.28		5.50	5.52	2.91	3.05	0.68	1.10
120		0.96	0.97	0.20	0.22	0.00	0.01		2.88	2.90	1.02	1.10	0.06	0.14
Panel C: Debt/Equity Ratio b = 2														
	T = 0.25							T = .5						
	a = 1	a = .75		a = .5		a = .25		a = 1	a = .75		a = .5		a = .25	
X		Φ_2=0	Φ_2=.05	Φ_2=0	Φ_2=.05	Φ_2=0	Φ_2=.05		Φ_2=0	Φ_2=.05	Φ_2=0	Φ_2=.05	Φ_2=0	Φ_2=.05
80		22.85	22.87	21.39	21.47	20.99	21.02		26.10	26.14	23.27	23.44	22.00	22.17
90		15.41	15.44	12.94	13.10	11.28	11.60		19.40	19.44	15.64	15.89	12.70	13.38
100		9.58	9.62	6.60	6.78	3.63	4.38		13.89	13.93	9.68	9.94	5.51	6.58
110		5.47	5.49	2.78	2.92	0.55	0.97		9.57	9.61	5.50	5.73	1.71	2.53
120		2.86	2.88	0.96	1.04	0.04	0.12		6.36	6.39	2.88	3.05	0.39	0.75

Table 2
Implied Volatilities
S = $100, r = 0.05, ó$_1$ = 0.2.

	T = 0.25							T = .5						
Panel A: Debt/Equity Ratio b = 0														
	a = 1	a = .75		a = .5		a = .25		a = 1	a = .75		a = .5		a = .25	
X	Black-Scholes	Φ_2=0	Φ_2=.05	Φ_2=0	Φ_2=.05	Φ_2=0	Φ_2=.05	Black-Scholes	Φ_2=0	Φ_2=.05	Φ_2=0	Φ_2=.05	Φ_2=0	Φ_2=.05
80	.200	.144	.144	.081	.081	.081	.235	.200	.143	.144	.081	.081	.081	.216
90	.200	.147	.148	.094	.098	.054	.054	.200	.147	.147	.093	.098	.037	.059
100	.200	.150	.150	.099	.103	.049	.062	.200	.149	.150	.099	.102	.048	.062
110	.200	.152	.152	.104	.106	.056	.065	.200	.152	.152	.103	.106	.055	.064
120	.200	.154	.154	---	---	---	---	.200	.154	.154	.107	.109	.060	.067
Panel B: Debt/Equity Ratio b = 1														
	T = 0.25							T = .5						
	a = 1	a = .75		a = .5		a = .25		a = 1	a = .75		a = .5		a = .25	
X		Φ_2=0	Φ_2=.05	Φ_2=0	Φ_2=.05	Φ_2=0	Φ_2=.05		Φ_2=0	Φ_2=.05	Φ_2=0	Φ_2=.05	Φ_2=0	Φ_2=.05
80		.313	.314	.200	.210	.081	.125		.314	.315	.200	.210	.081	.128
90		.306	.307	.200	.208	.094	.126		.307	.308	.200	.208	.093	.126
100		.301	.302	.200	.206	.099	.125		.302	.303	.200	.206	.099	.125
110		.296	.297	.200	.205	.104	.125		.297	.298	.200	.205	.103	.125
120		.292	.293	.200	.204	.108	.125		.293	.294	.200	.205	.107	.125
Panel C: Debt/Equity Ratio b = 2														
	T = 0.25							T = .5						
	a = 1	a = .75		a = .5		a = .25		a = 1	a = .75		a = .5		a = .25	
X		Φ_2=0	Φ_2=.05	Φ_2=0	Φ_2=.05	Φ_2=0	Φ_2=.05		Φ_2=0	Φ_2=.05	Φ_2=0	Φ_2=.05	Φ_2=0	Φ_2=.05
80		.482	.484	.313	.325	.144	.198		.485	.487	.314	.327	.143	.198
90		.466	.468	.306	.317	.147	.192		.469	.470	.307	.318	.147	.193
100		.452	.454	.301	.310	.150	.188		.455	.456	.302	.311	.149	.189
110		.441	.442	.296	.304	.152	.185		.443	.444	.297	.305	.152	.185
120		.430	.432	.292	.299	.154	.182		.432	.434	.293	.300	.154	.182

Table 3
Call price with X=425 resulting from modifying a and b

sigma1	1-a	b=0.05	b=0.10	b=0.15	b=0.20	b=0.25	b=0.30	b=0.35	b=0.40	b=0.45	b=0.50
0.155836	0.05	28.819	29.607	30.4018	31.2028	32.009	32.82	33.6352	34.4541	35.2764	36.1017
0.16446	0.1	29.1115	29.9229	30.7411	31.5652	32.3944	33.2281	34.066	34.9074	35.752	36.5995
0.174073	0.15	29.7037	30.5487	31.4	32.2568	33.1185	33.9845	34.8544	35.7276	36.6039	37.483
0.184862	0.2	30.4441	31.3206	32.2032	33.0912	33.9838	34.8806	35.7811	36.685	37.5917	38.5012
0.197062	0.25	31.0992	31.9975	32.9018	33.8116	34.726	35.6446	36.5669	37.4926	38.4212	39.3525
0.210975	0.3	31.5285	32.4376	33.353	34.2738	35.1995	36.1295	37.0633	38.0005	38.9407	39.8837
0.226996	0.35	31.7111	32.6226	33.5405	34.4641	35.3927	36.3257	37.2627	38.2032	39.1469	40.0933
0.245653	0.4	31.6931	32.6015	33.5166	34.4376	35.3637	36.2944	37.2291	38.1675	39.1091	40.0536
0.267663	0.45	31.5352	32.4376	33.3469	34.2622	35.1827	36.1079	37.0373	37.9703	38.9066	39.846
0.294034	0.5	31.2861	32.181	33.0829	33.9909	34.9043	35.8224	36.7447	37.6708	38.6002	39.5327

Table 4
Risk parameters corresponding to a fixed call price (27.904)

Table 4A Total risk (sigma) corresponding to a fixed call price (27.904)

1-a	b=0.05	b=0.10	b=0.15	b=0.20	b=0.25	b=0.30	b=0.35	b=0.40	b=0.45	b=0.50
0.05	0.1417095	0.1368285	0.1323609	0.1282538	0.1244627	0.120951	0.1176867	0.114643	0.1117965	0.1091273
0.1	0.13941843	0.1342765	0.12953055	0.1251276	0.1210233	0.1171807	0.113568	0.11015772	0.1069267	0.1038541
0.15	0.1340202	0.1282452	0.12283028	0.11772178	0.1128756	0.1082543	0.1038274	0.0995701	0.0954638	0.0914964
0.2	0.1260547	0.1195497	0.1134454	0.10772	0.1023668	0.09739	0.0927965	0.0885905	0.0847661	0.081306
0.25	0.1190916	0.1127097	0.1069549	0.1017907	0.0971705	0.0930393	0.089339	0.0860128	0.0830085	0.0802802
0.3	0.1162309	0.1105465	0.1055062	0.1010202	0.0970069	0.0933951	0.0901242	0.0871438	0.0844124	0.0818956
0.35	0.1164108	0.1111936	0.1065445	0.1023733	0.0986051	0.0951789	0.0920452	0.0891634	0.0865007	0.0840298
0.4	0.117971	0.112968	0.1084782	0.1044209	0.1007306	0.0973548	0.0942507	0.0913836	0.0887248	0.0862505
0.45	0.1201093	0.1151801	0.1107354	0.1067013	0.10301845	0.0996391	0.0965242	0.0936414	0.0909642	0.0884701
0.5	0.122515	0.1175853	0.1131282	0.10907368	0.1053657	0.1019586	0.0988147	0.0959031	0.0931976	0.0906762

Table 4B High-risk (sigma1) corresponding to a fixed call price (27.904)

1-a	b=0.05	b=0.10	b=0.15	b=0.20	b=0.25	b=0.30	b=0.35	b=0.40	b=0.45	b=0.50
0.05	0.14915652	0.14401822	0.13931509	0.13499143	0.13100042	0.12730352	0.12386704	0.120662782	0.1176661	0.114856071
0.1	0.15486055	0.149145419	0.14387028	0.13897627	0.13441409	0.13014269	0.12612673	0.122335671	0.1187438	0.115327896
0.15	0.1575498	0.150750229	0.14437415	0.13835842	0.13265111	0.12720814	0.12199363	0.116978348	0.1121404	0.107465477
0.2	0.15732524	0.149180735	0.14153654	0.1343654	0.12765898	0.12142264	0.11566513	0.110391892	0.1055957	0.101255136
0.25	0.15835963	0.149826056	0.14212851	0.13521857	0.12903432	0.12350258	0.11854596	0.114088771	0.1100614	0.106402573
0.3	0.16536522	0.157209588	0.14997488	0.14353291	0.1377671	0.1325757	0.12787208	0.12358416	0.1196526	0.11602814
0.35	0.17809918	0.170025784	0.16282759	0.15636575	0.15052495	0.14521122	0.14034831	0.135873674	0.1317368	0.127895527
0.4	0.1952289	0.186828568	0.179285	0.17246356	0.16625494	0.16057152	0.15534187	0.15050805	0.1460222	0.141844484
0.45	0.21649485	0.207451223	0.1992902	0.19187724	0.18510429	0.1788844	0.17314646	0.167831569	0.1628915	0.158285192
0.5	0.24251742	0.232551524	0.2235329	0.21532132	0.20780453	0.20089108	0.19450547	0.188585838	0.1830797	0.177942949

Table 5
Corresponding call prices for alternative exercise prices

Table 5A Call price with X = 400 for given parameters (including varying sigma2 as in Table 3B)

1-a	b=0.05	B=0.10	b=0.15	b=0.20	b=0.25	b=0.30	b=0.35	b=0.40	b=0.45	b=0.50
0.05	43.9468887	43.82001167	43.6944018	43.5701	43.447009	43.3252931	43.2048767	43.08578938	42.967991	42.85146786
0.1	43.6872987	43.53568939	43.3855184	43.2369665	43.0902036	42.9454667	42.8029386	42.66277227	42.525227	42.39045765
0.15	43.5218235	43.38498671	43.2571814	43.139937	43.0350953	42.9447025	42.8711978	42.81735588	42.786346	42.78163873
0.2	43.8016704	43.78198397	43.7948936	43.8429475	43.9270083	44.0454871	44.1937876	44.36465146	44.549058	44.73769371
0.25	44.5658258	44.68822838	44.8308263	44.9839055	45.1378442	45.2844782	45.4178691	45.53438192	45.632361	45.71164688
0.3	45.2577882	45.37925137	45.4894535	45.5838251	45.6601711	45.7181446	45.7586389	45.78329694	45.794126	45.79319018
0.35	45.5637397	45.62431488	45.6671682	45.6930164	45.7034837	45.700661	45.6867647	45.66387764	45.633907	45.59845129
0.4	45.5866747	45.59686162	45.5927376	45.5764749	45.5502793	45.5162471	45.476192	45.43165678	45.383873	45.33380909
0.45	45.4610378	45.43974869	45.4084564	45.369265	45.3240438	45.2743871	45.2215694	45.16655788	45.110131	45.05284032
0.5	45.2607945	45.22130775	45.1749163	45.1233337	45.0680102	45.0100868	44.9504152	44.88967044	44.828311	44.7666858

Table 5A Call price with X = 450 for given parameters (including varying sigma2 as in Table 3B)

1-a	b=0.05	B=0.10	b=0.15	b=0.20	b=0.25	b=0.30	b=0.35	b=0.40	b=0.45	b=0.50
0.05	16.297805	16.36931637	16.4410441	16.5128018	16.584393	16.6558116	16.7268986	16.79761689	16.867871	16.93764389
0.1	16.4579208	16.53701082	16.6153267	16.6925003	16.7681978	16.8421903	16.9142019	16.98392858	17.051172	17.11560367
0.15	16.4277062	16.4719695	16.5082895	16.5353256	16.5517509	16.5560379	16.5466403	16.52191171	16.480274	16.42036634
0.2	16.0557783	16.00548029	15.9363658	15.849003	15.7455037	15.6297676	15.5070001	15.38329376	15.264362	15.15478176
0.25	15.5189085	15.41157802	15.3044476	15.2031747	15.1121506	15.0339931	14.9697162	14.91906734	14.880992	14.85404463
0.3	15.2064621	15.13128069	15.0697809	15.0221749	14.9876193	14.9646846	14.9516749	14.94691963	14.9489	14.95622675
0.35	15.1595387	15.1239644	15.1007805	15.0882808	15.0846225	15.0881199	15.0973166	15.11092323	15.127953	15.14754123
0.4	15.2549481	15.24314135	15.240464	15.2451719	15.2556627	15.2706581	15.2890784	15.31011192	15.333103	15.35755667
0.45	15.4175051	15.41776904	15.4249321	15.4374766	15.4541594	15.4740321	15.4963371	15.52045135	15.545978	15.57257657
0.5	15.6169278	15.62237259	15.6335053	15.6490287	15.6679712	15.689558	15.7131548	15.73834231	15.764746	15.79211183

[1] A more general solution is

$$h_1 = (r - q_1)U,$$
$$h_2 = (r - q_2)V, \qquad (17^*)$$

where the arbitrary q_1 and q_2 bear the interpretation as the “hypothetical dividend yields” that would enable assets *U* and *V* to be treated as if they are tradable assets in a risk-neutral world (see Hull (1989), Appendix 7D for details).

Equation (18) is then replaced by

$$C_t + (r - q_1)UC_U + (r - q_2)VC_V + rLC_L$$
$$+ \tfrac{1}{2} s_1^2 U^2 C_{UU} + \tfrac{1}{2} s_2^2 V^2 C_{VV} + r\, s_1 s_2 UVC_{UV} - rC = 0, \qquad (18^*)$$

which is identical to the last equation of Hull (1989, Appendix 7D, p. 180). However, the two dividend yields are not entirely arbitrary, because the third equation of (16), resulting from the aggregate tradability, restricts q_1 and q_2 such that

$$q_1 U + q_2 V = 0\,. \qquad (19^*)$$

Since *U* and *V* are both positive, q_1 and q_2 must assume opposite signs. This implies that, if one risky asset needs to pay a positive dividend to enter the risk-neutral world, the other asset must pay a negative dividend. No economic justification for this implication is apparent. Besides, even if a negative dividend is justifiable, it is impossible to find a pair of dividend yield functions to satisfy (19*), unless

$$q_1 = -q_2 \frac{V}{U}; \qquad (20^*)$$

Such a restriction on q_1 based on *V* has no justification; hence rather than choosing a pair of non-zero dividend yields to satisfy (19*), it seems natural to set them both to zero.

[2] In general, for the a values used, the results were relatively insensitive to Φ_2

[3] The X=375 data point is anomalous; the market price (in fact, even the asked price) are infeasible, with a negative implied volatility, and at least $.50 increase in market price is required, thus eliminating the left extreme over-pricing.

Figure 1. (S&P)The pricing error is defined as the difference between the modelled price and the market bid-asked mid-point for both models; sigma is .0885 compared to the B-S implied volatility of .1204 (for X= 425), and the market prices range from $69 to $19.

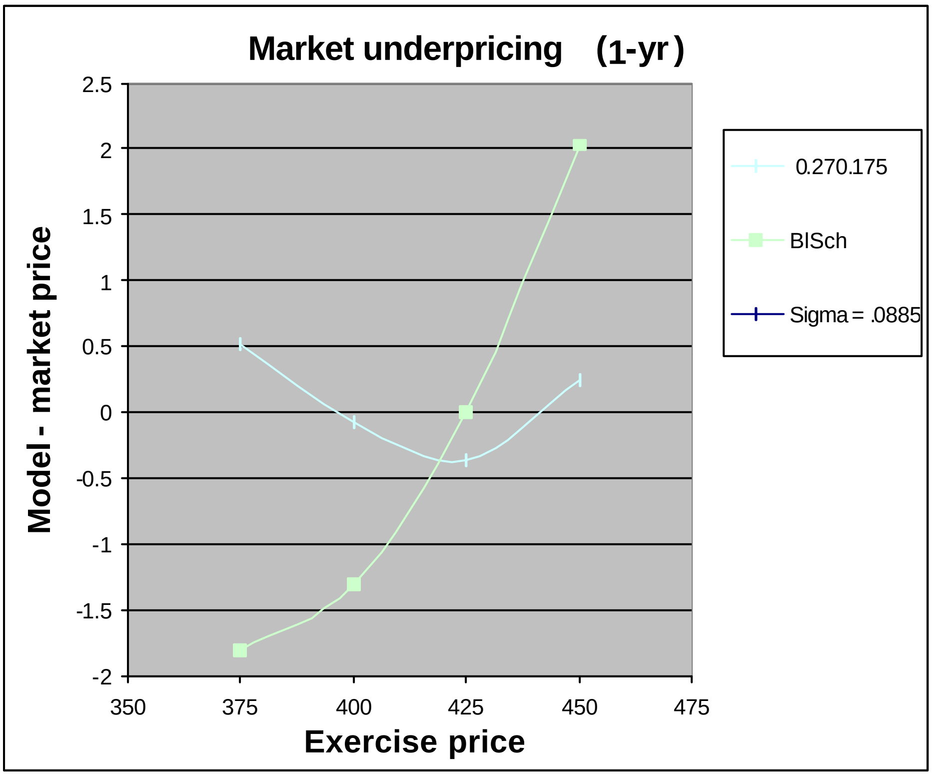


Figure 2.(S&P) The pricing error is defined as the difference between the modelled price and the market bid-asked mid-point for both models; sigma is .0885 compared to the B-S implied volatility of .1191 (for X= 425), and the market prices range from $36 to $14.

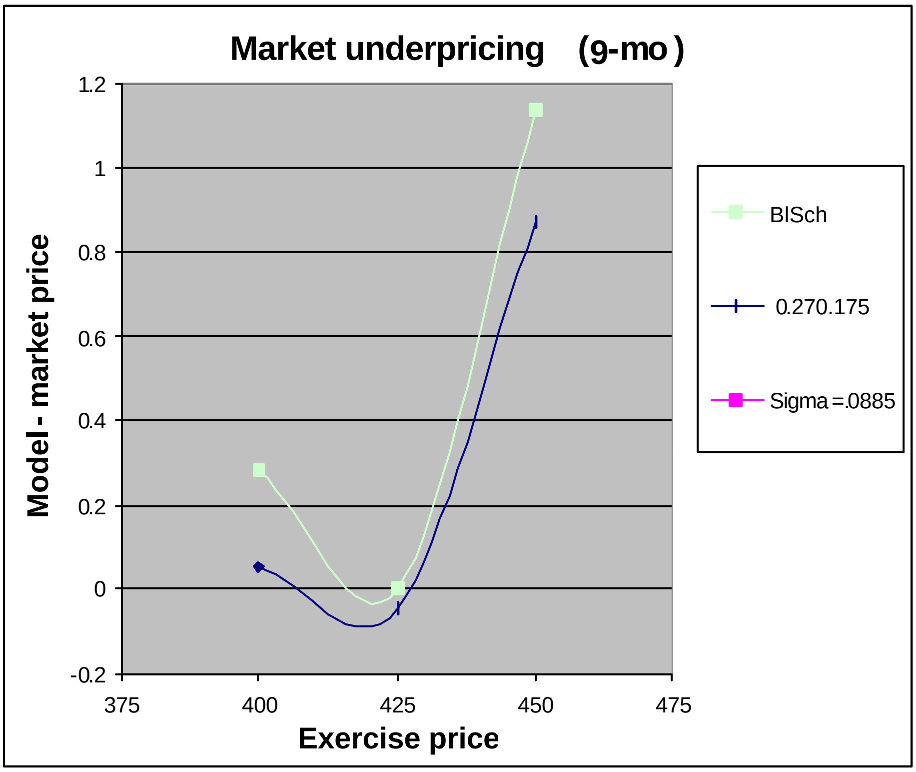


Figure 3. (S&P) The pricing error is defined as the difference between the modelled price and the market bid-asked mid-point for both models; sigma is .0830 compared to the B-S implied volatility of .1148 (for X= 435), and the market prices range from $65 to $5.

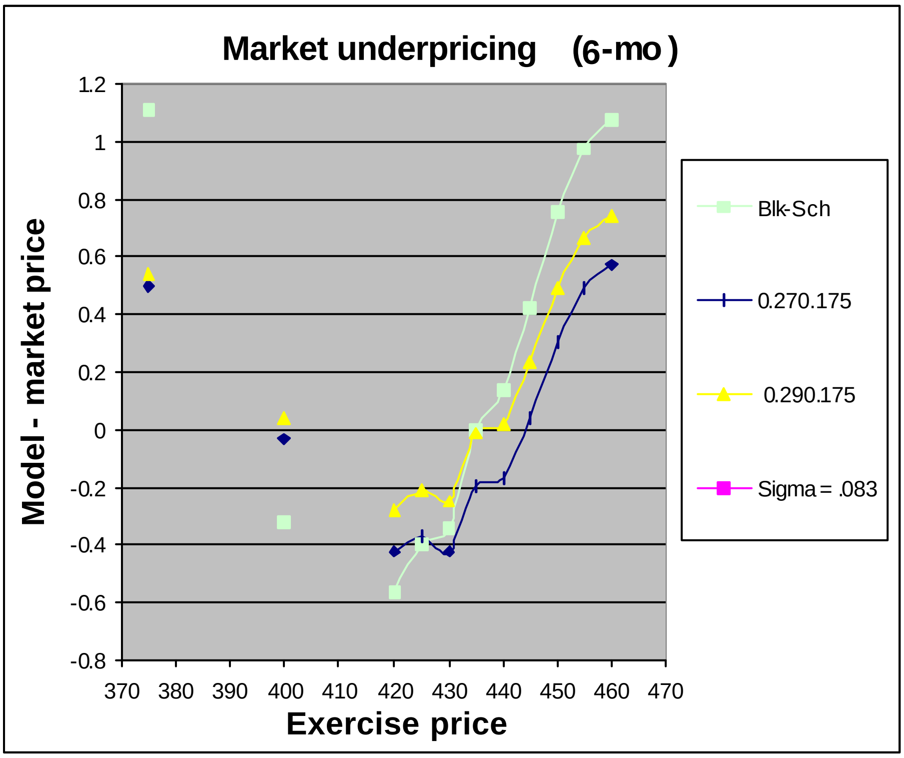


Figure 4. (S&P) The pricing error is defined as the difference between the modelled price and the market bid-asked mid-point for both models; sigma is .0875 compared to the B-S implied volatility of .1209 (for X= 425), and the market prices range from $63 to $1.50.

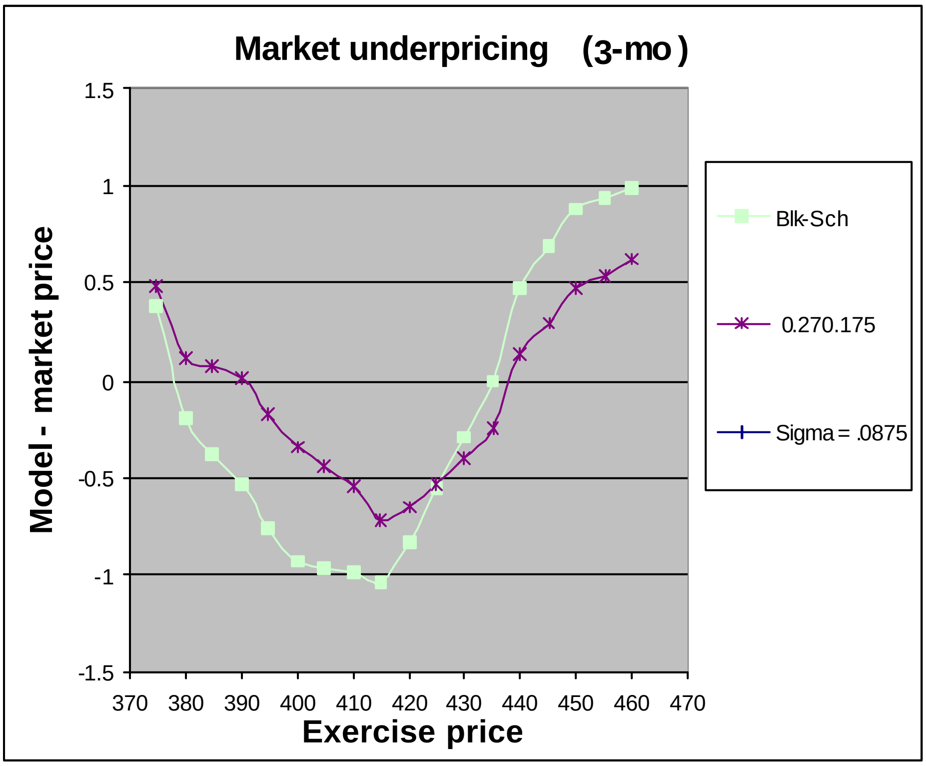


Figure 5. Corresponding impled volatilities throuh B-S by market and model prices for 3-mo. options

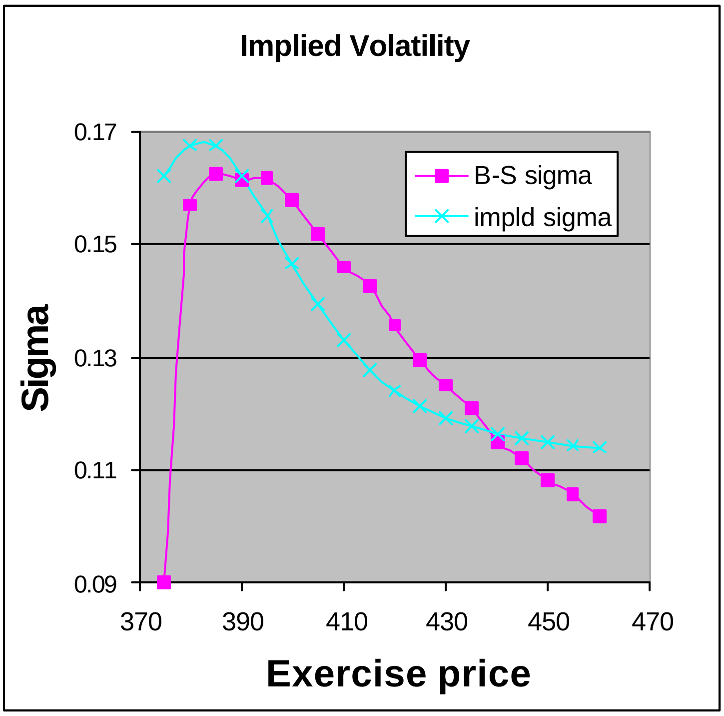
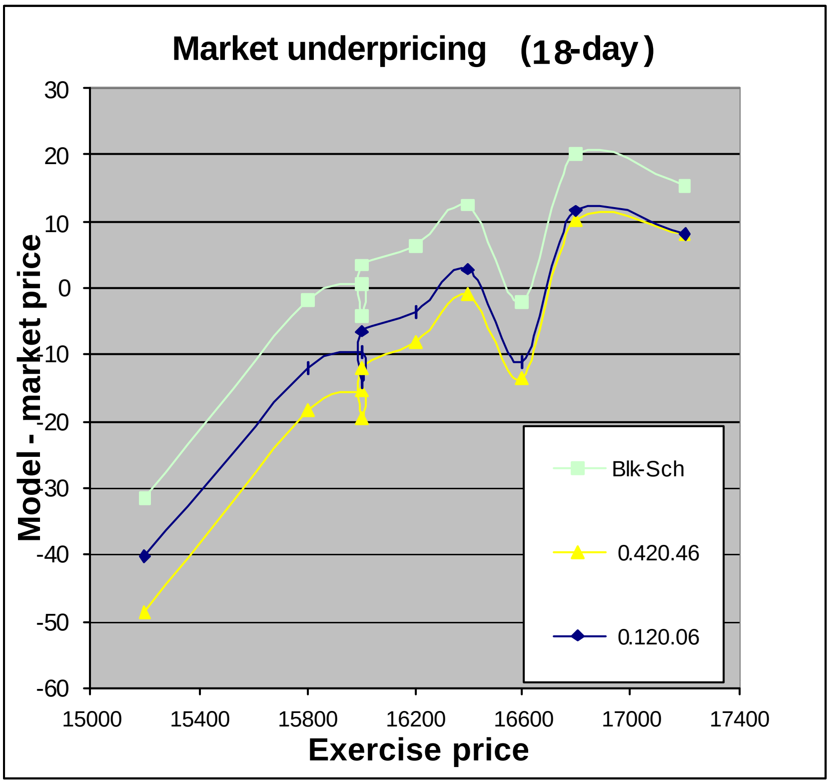


Table 6. Corresponding impled volatilities throuh B-S by market and model prices for 3-mo. options

X	Mkt Pr	B-S sigma	GR Pr	impld sigma
375	63.125	0.09	63.6066	0.16224
380	58.75	0.1568	58.86185	0.1676
385	54	0.16246	54.0778	0.1675
390	49.25	0.1614	49.26684	0.16226
395	44.625	0.16182	44.45053	0.1548
400	40	0.15815	39.65987	0.1468
405	35.375	0.1517	34.93586	0.1393

410	30.875	0.14594	30.32971	0.13295
415	26.625	0.14245	25.90171	0.12793
420	22.375	0.13555	21.71799	0.12415
425	18.375	0.12945	17.84508	0.12135
430	14.75	0.1249	14.34297	0.1193
435	11.5	0.1209	11.25783	0.1178
440	8.48	0.1149	8.616202	0.1166
445	6.125	0.11185	6.421871	0.11575
450	4.185	0.10834	4.656112	0.11506
455	2.75	0.1058	3.281147	0.1145
460	1.625	0.1017	2.245854	0.11408

Figure 6. (HSI) The pricing error is defined as the difference between the modelled price and the market bid-asked mid-point for both models; sigma is .36 or .28 compared to the B-S implied volatility of .40904 (for X= 16000), and the market prices range from 1000 to 150.


Market underpricing (18-day)
30
20
10
0
-10
-20
-30
-40
-50
-60
Model - market price
15000
15400
15800
16200
16600
17000
17400
Exercise price
Blk-Sch
0.420.46
0.120.06